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Perturbative dynamics of fuzzy spheres at large N

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Abstract: We clarify some peculiar aspects of the perturbative expansion around a classical fuzzy-sphere solution in matrix models with a cubic term. While the effective action in the large-N limit is saturated at the one-loop level, we find that the "one-loop dominance" does not hold for generic observables due to one-particle reducible diagrams. However, we may exploit the one-loop dominance for the effective action and obtain various observables to all orders from one-loop calculation by simply shifting the center of expansion to the "quantum solution", which extremizes the effective action. We confirm the validity of this method by comparison with the direct two-loop calculation and with Monte Carlo results in the 3d Yang-Mills-Chern-Simons matrix model. From the all order result we find that the perturbative expansion has a finite radius of convergence.

Keywords: Non-Commutative Geometry, Matrix Models.

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1. Introduction

Fuzzy spheres [1], which are simple compact noncommutative manifolds, have been studied extensively. There are various motivations for studying the fuzzy spheres. First it is expected that the noncommutative geometry provides a crucial link to string theory and quantum gravity. Indeed the Yang-Mills theory on noncommutative geometry is shown to emerge from a certain low-energy limit of string theory [2]. There is also an independent observation that the space-time uncertainty relation, which is naturally realized by noncommutative geometry, can be derived from some general assumptions on the underlying theory of quantum gravity [3]. One may also use fuzzy spheres as a regularization scheme alternative to the lattice regularization [4]. Unlike the lattice, fuzzy spheres preserve the continuous symmetries of the space-time considered, and the well-known problem of chiral symmetry [5 – 19] and supersymmetry in lattice theories may become easier to overcome.

As expected from the so-called Myers effect [20] in string theory, fuzzy spheres appear as classical solutions in matrix models with a Chern-Simons term [21 – 24]. The properties of the fuzzy spheres in matrix models have been studied in refs. [25 – 32]. One can actually use matrix models to define a regularized field theory on fuzzy spheres. Such an approach has been successful in the case of noncommutative torus [33], where nonperturbative studies have produced various important results [12, 34, 16]. These matrix models belong to the class of so-called large-N reduced models, which are believed to provide a constructive definition of superstring and M theories [35 – 37]. For instance the IIB matrix model [36], which can be obtained by dimensional reduction of ten-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory, is proposed as a constructive definition of type-IIB superstring theory. In this model the space-time is represented by the eigenvalues of bosonic matrices, and hence treated as a dynamical object. The dynamical generation of four-dimensional space-time has been discussed in refs. [38 – 51].

In ref. [52] we have performed a first nonperturbative study on the dynamical properties of the fuzzy spheres, which appear in a simple matrix model. The model can be obtained by dimensional reduction of 3d Yang-Mills-Chern-Simons (YMCS) theory, and it incorporates various fuzzy S^2 solutions. The most important non-perturbative result was that there exists a first-order phase transition as we vary the coefficient of the Chern-Simons term (α). In the small-α phase, the effect of the Chern-Simons term is negligible and the model behaves almost as the pure Yang-Mills model ($\alpha = 0$) [53]. In the large-α phase, on the other hand, a single fuzzy sphere appears dynamically.[1] In this phase Monte Carlo data agree very well with the one-loop results, which led us to speculate that the one-loop contribution dominates the quantum correction in the large-N limit [52].

In the present paper we first address this issue by direct two-loop calculation. While the "one-loop dominance" does hold for the effective action [27, 31, 32], we find that this is not the case for generic observables. The higher-loop contribution that survives the large-N limit actually comes from one-particle reducible diagrams, which do not appear in the calculation of the effective action. However, we may exploit the one-loop dominance for the effective action and obtain various observables *to all orders* from one-loop calculation by simply shifting the center of expansion to the "quantum solution", which extremizes the effective action. We confirm the validity of this method proposed by Kitazawa et al. [31] by comparison with the direct two-loop calculation and with Monte Carlo results.

From the all order result we find that the perturbative expansion has a finite radius of convergence, and the lower critical point of the first-order phase transition lies precisely on the convergence circle. We also reconsider the issue of the dynamical gauge group, which was previously discussed at the one-loop level [52]. The all order calculation of the free energy for k coinciding fuzzy spheres allows us to obtain a more definite conclusion on this issue.

This paper is organized as follows. In section 2 we define the model and briefly review the results obtained in our previous work. In section 3 we perform explicit two-loop calculation of an observable and demonstrate that the two-loop contribution survives the large-N limit. In section 4 we obtain an all order result from one-loop calculation by shifting the center of expansion to the "quantum solution". In section 5 we extend the all order calculation to more general observables. In section 6 we compare the results of the all order calculation with our Monte Carlo results. In section 7 we address the issue of the dynamical gauge group using the all order result for the free energy. Section 8 is devoted to a summary and concluding remarks. In appendix A we give the details of the two-loop calculation.

[1]This work has been extended to matrix models which incorporate four-dimensional fuzzy manifolds as classical solutions [54 – 56]. While the fuzzy S^4 turned out to be always unstable, the fuzzy CP^2 and the fuzzy $S^2 \times S^2$ can be stabilized in the large-N limit.

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2. Brief review of the model

The model we study in this paper is given by the action

$$S[A] = N \,\mathrm{tr} \left(-\frac{1}{4} [A_\mu, A_\nu]^2 + \frac{2}{3} \, i \, \alpha \, \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \right) , \tag{2.1}$$

where A_μ ($\mu = 1, 2, 3$) are $N \times N$ traceless hermitian matrices. Here and henceforth we assume that summation is taken over repeated Greek indices. The classical equation of motion is given by

$$[A_\mu, [A_\mu, A_\nu]] + i \, \alpha \, \epsilon_{\nu\sigma\rho} [A_\sigma, A_\rho] = 0 \, . \tag{2.2}$$

As a solution, we consider

$$A_\mu = X_\mu \stackrel{\mathrm{def}}{=} \alpha \, L_\mu \, , \tag{2.3}$$

where L_μ is the N-dimensional irreducible representation of the SU(2) Lie algebra

$$[L_\mu, L_\nu] = i \, \epsilon_{\mu\nu\rho} \, L_\rho \, . \tag{2.4}$$

This solution corresponds to the fuzzy S^2, and since

$$(X_\mu)^2 = \frac{1}{4} \, \alpha^2 \, (N^2 - 1) \, \mathbf{1}_N \, , \tag{2.5}$$

the radius of the sphere is given by $R = \frac{1}{2} \alpha \sqrt{N^2 - 1}$.

In ref. [52] we found that the system undergoes a first-order phase transition as we vary α. In the large-α regime the dominant configurations are close to the fuzzy sphere (2.3), while in the small-α regime the large-N property is similar to the pure Yang-Mills model ($\alpha = 0$) [53], and the geometry of the dominant configurations is given by that of a solid ball. The two phases are called the "fuzzy sphere phase" and the "Yang-Mills phase", respectively. When we discuss the large-N limit in the fuzzy sphere phase, the natural parameter to fix turned out to be

$$\tilde{\alpha} = \alpha \sqrt{N} \, . \tag{2.6}$$

The lower critical point obtained from Monte Carlo simulation is

$$\tilde{\alpha}_{\mathrm{cr}} \simeq 2.1 \, , \tag{2.7}$$

which was reproduced later from the one-loop effective action as[2]

$$\tilde{\alpha}_{\mathrm{cr}} = \sqrt[4]{\frac{512}{27}} = 2.0867794 \cdots \, . \tag{2.8}$$

In the fuzzy sphere phase, various observables agree well with the one-loop calculation. We therefore speculated that the one-loop dominance, which was previously claimed for the effective action [27], holds also for observables.

[2] This analytical result was informed to us by D. O'Connor after J.N. gave a seminar on the Monte Carlo results including (2.7) at the Dublin Institute for Advanced Studies (DIAS). Its derivation in section 4 is due to Y. Kitazawa (private communication).

3. Explicit two-loop calculation

In order to see whether the one-loop dominance holds also for observables, we perform explicit two-loop calculation around the fuzzy sphere solution. We decompose A_μ into the classical background $X_\mu = \alpha L_\mu$ and the fluctuation $\tilde{A}_\mu$ as

$$A_\mu = X_\mu + \tilde{A}_\mu \,. \tag{3.1}$$

We introduce the following gauge fixing term and the corresponding ghost term

$$S_{\text{g.f.}} = -\frac{1}{2} N \,\text{tr}\Big([X_\mu, A_\mu]^2\Big) \,, \tag{3.2}$$

$$S_{\text{gh}} = -N \,\text{tr}([X_\mu, \bar{c}][A_\mu, c]) \,. \tag{3.3}$$

The total action can be written as

$$S_{\text{total}} = S + S_{\text{g.f.}} + S_{\text{gh}} \tag{3.4}$$

$$= S[X] + S_{\text{kin}} + S_{\text{int}} \,, \tag{3.5}$$

where the kinetic term S_{kin} and the interaction term S_{int} are given by

$$S_{\text{kin}} = \frac{1}{2} N \,\text{tr}(\tilde{A}_\mu[X_\lambda, [X_\lambda, \tilde{A}_\mu]]) + N \,\text{tr}(\bar{c}[X_\lambda, [X_\lambda, c]]) \,, \tag{3.6}$$

$$S_{\text{int}} = -\frac{1}{4} N \,\text{tr}([\tilde{A}_\mu, \tilde{A}_\nu]^2) - N \,\text{tr}([\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu]) + \\ + \frac{2}{3} i \,\alpha \, N \,\epsilon_{\mu\nu\rho} \,\text{tr}(\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho) - N \,\text{tr}([X_\mu, \bar{c}][\tilde{A}_\mu, c]) \,. \tag{3.7}$$

The free energy W defined by

$$e^{-W} = \int d\tilde{A} \, dc \, d\bar{c} \, e^{-S_{\text{total}}} \tag{3.8}$$

can be calculated as a perturbative expansion

$$W = \sum_{j=0}^{\infty} W_j \,, \tag{3.9}$$

where W_j represents the j-th order contribution. The first two terms are obtained as [21, 52]

$$W_0 = S[X] = -\frac{\tilde{\alpha}^4}{24}(N^2 - 1) \,, \tag{3.10}$$

$$W_1 = \frac{1}{2} \sum_{l=1}^{N-1} (2l+1) \log\Big[\tilde{\alpha}^2 \, l \,(l+1)\Big] \,. \tag{3.11}$$

In order to calculate W_2, we have to evaluate the two-loop diagrams[3] depicted in figure 1. The solid line and the dashed line represent the propagators of $\tilde{A}_\mu$ and the ghost, respectively. The three-point vertices with (without) a dot represent the third (second) term in (3.7).

[3]The diagrams (a)–(d) are the same as the ones that appear in ref. [27]. The diagram (e) of ref. [27], which involves a fermion loop, does not appear in the present bosonic model.

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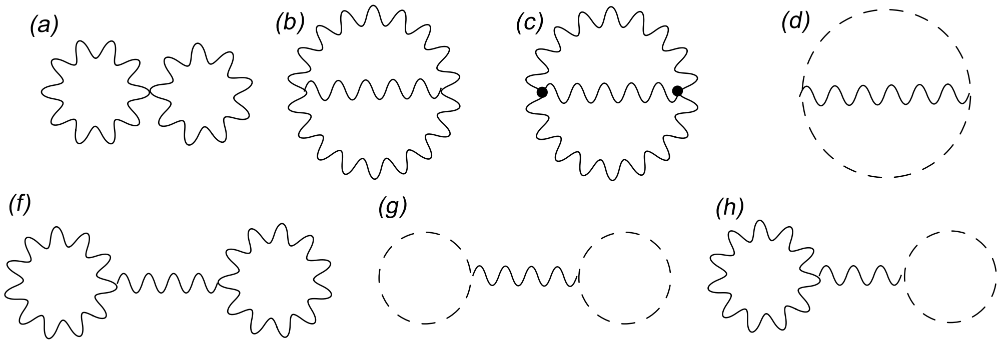

Figure 1: The two-loop diagrams that appear in the perturbative calculation of the free energy. The diagrams (a)–(d) are one-particle irreducible, while the rest are not.

As a fundamental observable, let us consider the vacuum expectation value of the action $\langle S \rangle$. The perturbative expansion of this quantity can be readily obtained from the results for the free energy (3.9). Let us define the rescaled action

$$S(\lambda, \alpha) = \lambda N \,\mathrm{tr} \left(-\frac{1}{4} [A_\mu, A_\nu]^2 + \frac{2}{3} i \, \alpha \, \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \right) , \tag{3.12}$$

and the corresponding free energy

$$e^{-W(\lambda,\alpha)} = \int dA \, dc \, d\bar{c} \, e^{-S(\lambda,\alpha)} , \tag{3.13}$$

which is related to the original free energy $W = W(1, \alpha)$ through

$$W(\lambda, \alpha) = \frac{3}{4} (N^2 - 1) \log \lambda + W(1, \lambda^{\frac{1}{4}} \alpha) . \tag{3.14}$$

Then the observable $\langle S \rangle$ can be written as

$$\begin{aligned} \frac{1}{N^2} \langle S \rangle &= \frac{1}{N^2} \left. \frac{\partial W(\lambda, \alpha)}{\partial \lambda} \right|_{\lambda=1} \\ &= \frac{3}{4} \left(1 - \frac{1}{N^2} \right) + \frac{\tilde{\alpha}}{4N^2} \frac{\partial W}{\partial \tilde{\alpha}} . \end{aligned} \tag{3.15}$$

Using eqs. (3.10) and (3.11), we obtain the one-loop result

$$\frac{1}{N^2} \langle S \rangle_{\text{1-loop}} = \left(-\frac{1}{24} \tilde{\alpha}^4 + 1 \right) \left(1 - \frac{1}{N^2} \right) . \tag{3.16}$$

The effective action can be obtained by restricting the diagrams that appear in the perturbative expansion of the free energy to one-particle irreducible (1PI) diagrams. It is the same as the free energy at the one-loop level, but starts to deviate at the two-loop level. In ref. [27] the two-loop effective action was calculated by evaluating the 1PI diagrams (a)–(d) in figure 1, and the two-loop effect turned out to vanish in the large-N limit. However, in order to calculate the observable $\langle S \rangle$, we need to evaluate the one-particle reducible (1PR) diagrams (f)–(h) in figure 1, which actually turn out to survive the large-N limit.

If we parametrize the higher order contributions W_j ($j \geq 2$) to the free energy in eq. (3.9) as

$$W_j = -N^2\, w_j(N)\, \tilde{\alpha}^{4(1-j)}\,, \tag{3.17}$$

the perturbative expansion of the observable $\langle S \rangle$ can be written as

$$\frac{1}{N^2}\langle S \rangle = \frac{1}{N^2}\langle S \rangle_{\text{1-loop}} + \sum_{j=2}^{\infty}(j-1)w_j(N)\, \tilde{\alpha}^{4(1-j)}\,. \tag{3.18}$$

The contribution of the 1PI diagrams to $w_2(N)$ is calculated as [27]

$$w_2^{(1\mathrm{PI})}(N) = \frac{1}{N}\left\{F_1(N) + 4\,F_3(N)\right\}, \tag{3.19}$$

where the functions $F_1(N)$ and $F_3(N)$ are defined by eqs. (A.20) and (A.22). The large-N behavior is found to be

$$w_2^{(1\mathrm{PI})}(N) \simeq \mathrm{O}\left(\frac{(\log N)^2}{N^2}\right), \tag{3.20}$$

which vanishes in the large-N limit.

On the other hand, the contribution of the 1PR diagrams to $w_2(N)$ are calculated as

$$w_2^{(1\mathrm{PR})}(N) = \frac{1}{2N}F_4(N)\,, \tag{3.21}$$

where $F_4(N)$ is given explicitly by eq. (A.28). In figure 2 we plot $w_2^{(1\mathrm{PR})}(N)$ against $1/N^2$. We find that

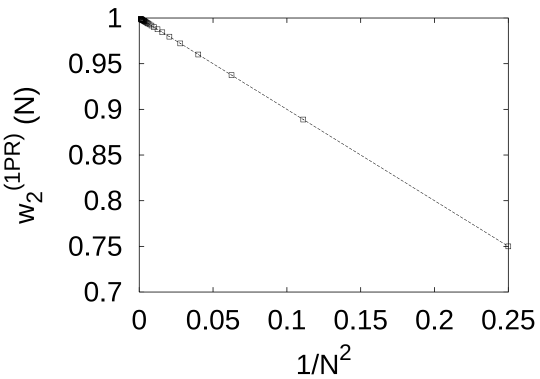

Figure 2: The coefficient $w_2^{(1\mathrm{PR})}(N)$, which comes from the 1PR two-loop diagrams, is plotted against $\frac{1}{N^2}$ for $N = 2, 3, 4, \ldots, 32$. The straight line represents $w_2^{(1\mathrm{PR})}(N) = 1 - \frac{1}{N^2}$.

$$w_2^{(1\mathrm{PR})}(N) = 1 - \frac{1}{N^2} \tag{3.22}$$

within the machine precision for $N = 2, 3, 4, \ldots, 32$. Thus the 1PR diagrams yield the contribution to $w_2(N)$ which survives the large-N limit. Summing the contributions from the 1PI diagrams and the 1PR diagrams, we obtain the large-N behavior

$$w_2(N) = w_2^{(1\mathrm{PI})}(N) + w_2^{(1\mathrm{PR})}(N) = 1 + \mathrm{O}\left(\frac{(\log N)^2}{N^2}\right). \tag{3.23}$$

4. All order result from one-loop calculation

In this section we apply a method in ref. [31] to obtain an all order result for the observable from one-loop calculation. The crucial point is that the free energy and the effective action are related to each other by the Legendre transformation. Therefore, one can obtain the free energy by evaluating the effective action at its extremum. Since the effective action enjoys the one-loop dominance in the case at hand, we may obtain the free energy, and hence the observable, to all orders in $\frac{1}{\tilde{\alpha}^4}$ in the large-N limit.

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By expanding the theory around a rescaled fuzzy-sphere configuration $A_\mu = \beta\, L_\mu$, we obtain the one-loop effective action Γ in the large-N limit as[4]

$$\lim_{N\to\infty} \frac{1}{N^2}\,\Gamma(\tilde\beta) = \left(\frac{1}{8}\,\tilde\beta^4 - \frac{1}{6}\,\tilde\alpha\,\tilde\beta^3\right) + \log\tilde\beta\,, \tag{4.1}$$

where $\tilde\beta = \beta\sqrt{N}$. The function of $\tilde\beta$ on the right-hand side has a local minimum for $\tilde\alpha > \sqrt[4]{512/27}$, from which one obtains the critical point $\tilde\alpha_{\rm cr}$ in eq. (2.8). The value of $\tilde\beta$ which gives the local minimum can be obtained by solving a fourth order algebraic equation, and it can be written explicitly as

$$\tilde\beta = f(\tilde\alpha) \stackrel{\rm def}{=} \frac{1}{4}\,\tilde\alpha\left(1+\sqrt{1+\delta}+\sqrt{2-\delta+\frac{2}{\sqrt{1+\delta}}}\right), \tag{4.2}$$

where

$$\delta = 4\,\tilde\alpha^{-\frac{4}{3}}\left[\left(1+\sqrt{1-\frac{512}{27\,\tilde\alpha^4}}\right)^{\frac{1}{3}} + \left(1-\sqrt{1-\frac{512}{27\,\tilde\alpha^4}}\right)^{\frac{1}{3}}\right]. \tag{4.3}$$

Plugging this solution into the one-loop effective action (4.1), we obtain an all order result for the free energy as

$$\lim_{N\to\infty} \frac{1}{N^2} W = \left(\frac{1}{8}\,f(\tilde\alpha)^4 - \frac{1}{6}\,\tilde\alpha\,f(\tilde\alpha)^3\right) + \log f(\tilde\alpha)\,. \tag{4.4}$$

By using (3.15), we can readily obtain an all order result for the observable $\langle S\rangle$ as

$$\lim_{N\to\infty} \frac{1}{N^2}\langle S\rangle = \frac{3}{4} - \frac{1}{24}\,\tilde\alpha\,f(\tilde\alpha)^3\,, \tag{4.5}$$

where we have used the fact that $\beta = f(\tilde\alpha)$ extremizes the one-loop effective action (4.1).

In order to check that the two-loop contribution obtained in section 3 can be reproduced correctly, let us expand the all order results at large $\tilde\alpha$. First the solution (4.2) can be expanded in terms of $\frac{1}{\tilde\alpha^4}$ as

$$f(\tilde\alpha) = \tilde\alpha\left(1-\sum_{j=1}^{\infty} c_j\,\tilde\alpha^{-4j}\right) = \tilde\alpha\left(1-\frac{2}{\tilde\alpha^4}-\frac{12}{\tilde\alpha^8}-\frac{120}{\tilde\alpha^{12}}-\frac{1456}{\tilde\alpha^{16}}-\cdots\right). \tag{4.6}$$

The expansions for the free energy and the observable are obtained respectively as

$$\lim_{N\to\infty} \frac{1}{N^2} W = -\frac{\tilde\alpha^4}{24} + \log\tilde\alpha - \frac{1}{\tilde\alpha^4} - \frac{14}{3\tilde\alpha^8} - \frac{110}{3\tilde\alpha^{12}} - \frac{364}{\tilde\alpha^{16}} - \cdots\,, \tag{4.7}$$

$$\lim_{N\to\infty} \frac{1}{N^2}\langle S\rangle = -\frac{\tilde\alpha^4}{24} + 1 + \frac{1}{\tilde\alpha^4} + \frac{28}{3\tilde\alpha^8} + \frac{110}{\tilde\alpha^{12}} + \frac{1456}{\tilde\alpha^{16}} + \cdots\,. \tag{4.8}$$

The third term, which corresponds to the two-loop contribution, indeed agrees with the result (3.23), which we obtained by the direct calculation.

[4] Here and henceforth we neglect an irrelevant constant term in the effective action and the free energy.

Let us discuss the convergence radius of these series expansions. In figure 3 we plot the ratio of the coefficients $\frac{c_{j+1}}{c_j}$ in (4.6), which is found to converge as

$$\lim_{j\to\infty} \frac{c_{j+1}}{c_j} = \frac{512}{27} = \tilde{\alpha}_{\rm cr}{}^4 \,. \tag{4.9}$$

This implies that the infinite series (4.6) converges for $\tilde{\alpha} > \tilde{\alpha}_{\rm cr}$. The expansion (4.7) for the free energy as well as that for the observable (4.8) has the same property. Thus in contrast to the perturbation theory in field theories, which usually yields merely an asymptotic expansion, the perturbative expansion around the fuzzy sphere in the matrix model has a finite radius of convergence. The lower critical point (2.8) lies precisely on the convergence circle.

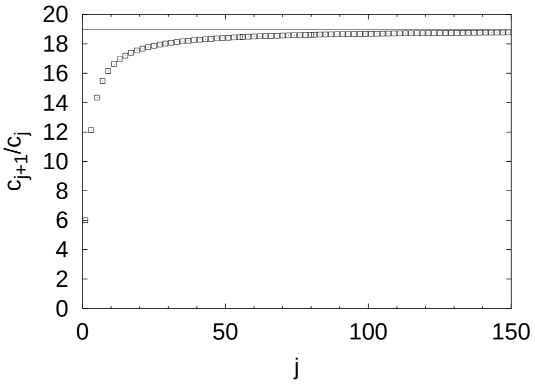

Figure 3: The ratio $\frac{c_{j+1}}{c_j}$ of the coefficients in (4.6). The horizontal line represents $\frac{512}{27} = 18.96296\ldots$

5. All order calculation of other observables

So far we have focused on a particular observable $\langle S \rangle$, which can be obtained by differentiating the free energy. However, the method for deriving all order results is applicable to general observables [31] since the calculation reduces to the evaluation of the free energy with an appropriate source term.

Suppose we want to calculate $\langle \mathcal{O} \rangle$. Then we consider an action

$$S_\epsilon = S + \epsilon\, \mathcal{O} \,, \tag{5.1}$$

and calculate the corresponding free energy W_ϵ, which has the expansion

$$W_\epsilon = W + \epsilon \, \langle \mathcal{O} \rangle + \mathrm{O}(\epsilon^2) \,. \tag{5.2}$$

In order to calculate the free energy to all orders, we first calculate the effective action for the system (5.1) around the rescaled configuration $A_\mu = \beta\, L_\mu$. Then the free energy W_ϵ can be obtained by evaluating the effective action at its extremum. Since the effective action enjoys the one-loop dominance, we obtain the free energy to all orders from the one-loop effective action. Let us expand the one-loop effective action Γ_ϵ as

$$\Gamma_\epsilon(\tilde{\beta}) = \Gamma(\tilde{\beta}) + \epsilon\, \Gamma_1(\tilde{\beta}) + \mathrm{O}(\epsilon^2) \,. \tag{5.3}$$

The "quantum solution" is given by solving the equation

$$\frac{\partial}{\partial \tilde{\beta}} \Gamma_\epsilon(\tilde{\beta}) = 0 \,, \tag{5.4}$$

whose solution is denoted as

$$\tilde{\beta} = f(\tilde{\alpha}) + \epsilon\, g(\tilde{\alpha}) + \mathrm{O}(\epsilon^2) \,. \tag{5.5}$$

Note that the first term is given by (4.2), and the second term represents a shift due to the source term.

By plugging this solution into (5.3) and extracting the $\mathrm{O}(\epsilon)$ term, we obtain the $\mathrm{O}(\epsilon)$ term of the free energy, which is nothing but $\langle \mathcal{O} \rangle$. Thus we obtain

$$\langle \mathcal{O} \rangle = \Gamma_1 \Big(f(\tilde{\alpha}) \Big) + g(\tilde{\alpha}) \left. \frac{\partial \Gamma}{\partial \tilde{\beta}} \right|_{\tilde{\beta} = f(\tilde{\alpha})} . \tag{5.6}$$

Note that the second term vanishes since $\tilde{\beta} = f(\tilde{\alpha})$ extremizes $\Gamma(\tilde{\beta})$, and the first term can be obtained by omitting the 1PR diagrams in the one-loop calculation of the observable $\langle \mathcal{O} \rangle$ and replacing $\tilde{\alpha}$ by $f(\tilde{\alpha})$.

As a concrete observable let us consider the spacetime extent $\left\langle \frac{1}{N} \operatorname{tr}(A_\mu)^2 \right\rangle$. The one-loop result is given as [52]

$$\lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{N} \operatorname{tr}(A_\mu)^2 \right\rangle_{\text{1-loop}} = \frac{1}{4}\,\tilde{\alpha}^2 - \frac{1}{\tilde{\alpha}^2} . \tag{5.7}$$

The first (second) term corresponds to the classical (one-loop) contribution. As can be seen from (C.8) of ref. [52], the one-loop contribution is given totally by a tadpole diagram, which is one-particle reducible. Therefore, the all order result can be obtained as

$$\lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{N} \operatorname{tr}(A_\mu)^2 \right\rangle = \frac{1}{4}\, f(\tilde{\alpha})^2 = \frac{1}{4}\,\tilde{\alpha}^2 - \frac{1}{\tilde{\alpha}^2} - \frac{5}{\tilde{\alpha}^6} - \frac{48}{\tilde{\alpha}^{10}} - \frac{572}{\tilde{\alpha}^{14}} - \cdots . \tag{5.8}$$

Next we calculate the Chern-Simons term

$$M = \frac{2\,i}{3\,N}\, \epsilon_{\mu\nu\rho} \operatorname{tr}(A_\mu A_\nu A_\rho) . \tag{5.9}$$

The one-loop result in the large-N limt is given as [52]

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \langle M \rangle_{\text{1-loop}} = -\frac{1}{6}\,\tilde{\alpha}^3 + \frac{1}{\tilde{\alpha}} , \tag{5.10}$$

where the first (second) term corresponds to the classical (one-loop) contribution. Similarly to the case of $\langle \frac{1}{N} \operatorname{tr} A^2 \rangle$, the one-loop term comes solely from 1PR diagrams. Thus we obtain the all order result as

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \langle M \rangle = -\frac{1}{6}\, f(\tilde{\alpha})^3 = -\frac{1}{6}\,\tilde{\alpha}^3 + \frac{1}{\tilde{\alpha}} + \frac{4}{\tilde{\alpha}^5} + \frac{112}{3\tilde{\alpha}^9} + \frac{440}{\tilde{\alpha}^{13}} + \cdots . \tag{5.11}$$

The all order result for $\langle \frac{1}{N} \operatorname{tr}(F_{\mu\nu})^2 \rangle$, where $F_{\mu\nu} = i\,[A_\mu, A_\nu]$, can be readily obtained by using the exact result [52]

$$\left\langle \frac{1}{N} \operatorname{tr}(F_{\mu\nu})^2 \right\rangle + 3\,\alpha \,\langle M \rangle = 3 \left(1 - \frac{1}{N^2} \right) . \tag{5.12}$$

Using (5.11), we obtain

$$\lim_{N\to\infty} \left\langle \frac{1}{N} \operatorname{tr}(F_{\mu\nu})^2 \right\rangle = 3 + \frac{1}{2}\,\tilde{\alpha}\, f(\tilde{\alpha})^3 = \frac{1}{2}\,\tilde{\alpha}^4 - \frac{12}{\tilde{\alpha}^4} - \frac{112}{\tilde{\alpha}^8} - \frac{1320}{\tilde{\alpha}^{12}} - \cdots . \tag{5.13}$$

From the relation

$$\frac{1}{N^2} \langle S \rangle = \frac{1}{4} \left\langle \frac{1}{N} \operatorname{tr}(F_{\mu\nu})^2 \right\rangle + \frac{1}{\sqrt{N}}\,\tilde{\alpha}\,\langle M \rangle , \tag{5.14}$$

we find that (5.11) and (5.13) are consistent with the result (4.5) obtained in the previous section. Clearly the series expansions that appear in this section have the same radius of convergence as that for $f(\tilde{\alpha})$.

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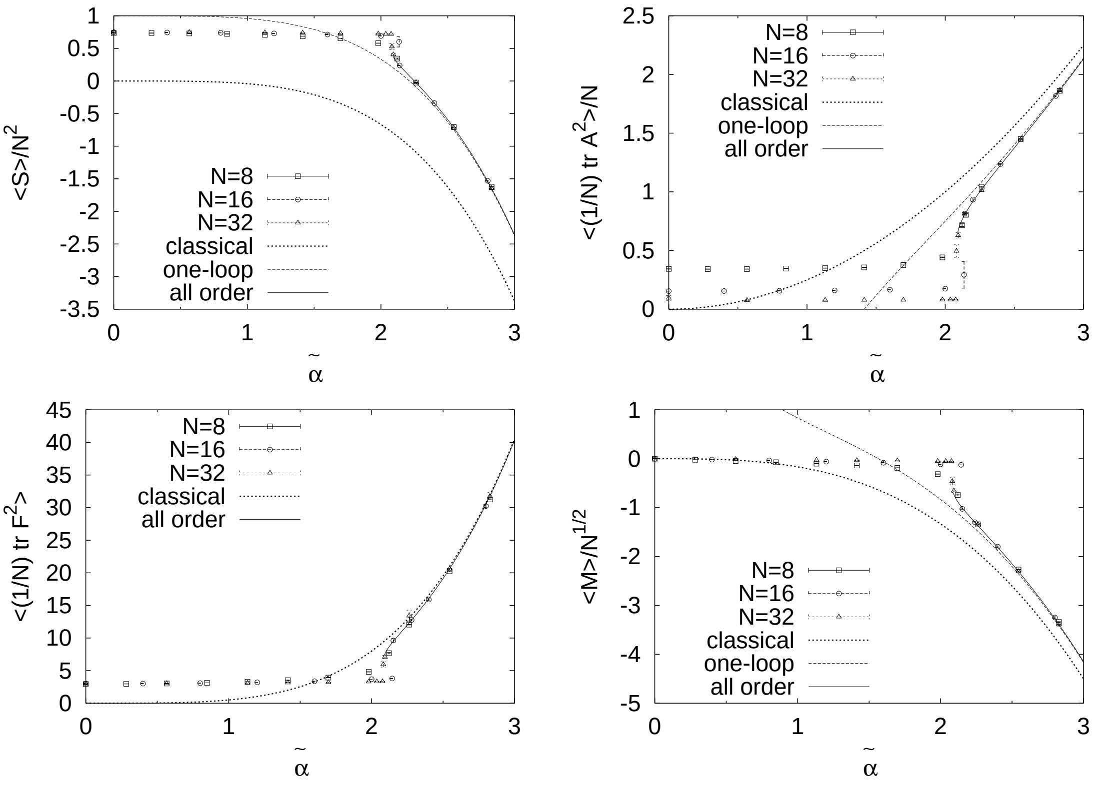

Figure 4: Monte Carlo results for various quantities are plotted against $\tilde{\alpha}$ for $N = 8, 16, 32$. The dotted (dashed) line represents the classical (one-loop) result, while the solid line represents the "all order" result. For the observable $\langle \frac{1}{N}\,\mathrm{tr}(F_{\mu\nu})^2 \rangle$, the one-loop result coincides with the classical result since there is no one-loop term; See eq. (5.13).

6. Comparison with Monte Carlo results

In this section we compare the all order results (4.5), (5.8), (5.11) and (5.13) obtained in the previous section with our Monte Carlo data in ref. [52]. Figure 4 shows the Monte Carlo results for the four observables as a function of $\tilde{\alpha}$ for $N = 8, 16, 32$, where we also plot the classical, one-loop, and all order results. Note that the lines representing the all order results terminate at the critical point $\tilde{\alpha} = \tilde{\alpha}_{\mathrm{cr}}$. The all order results nicely reproduce the behavior near the critical point. This reinforces the validity of the all order calculation.

7. The dynamical gauge group revisited

The 3d YMCS model (2.1) has various classical solutions, which represent multi fuzzy spheres. Among them, the solutions

$$A_\mu = \alpha\, L_\mu^{(n)} \otimes \mathbf{1}_k \tag{7.1}$$

with $N = n\,k$, which correspond to k-coincident fuzzy spheres, are of particular interest since they give rise to a noncommutative gauge theory on the fuzzy sphere with the gauge group of rank k [21]. If the true vacuum turns out to be described by the solution with some

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k, we may conclude that the gauge group of rank k has been generated dynamically. In ref. [52] we discussed this issue by comparing the one-loop effective action evaluated *at the classical solutions* (7.1). We found that the single fuzzy sphere always has the lowest free energy at the one-loop level in the fuzzy sphere phase. This implies that the dynamically generated gauge group is of rank one. From the view point of the all order calculation discussed in section 4, we should consider the rescaled configuration

$$A_\mu = \beta \, L_\mu^{(n)} \otimes \mathbf{1}_k \,, \tag{7.2}$$

and evaluate the one-loop effective action *at its extremum*, which gives the all order result for the free energy. In what follows we examine whether the higher order corrections to the free energy alter the conclusion.

The one-loop effective action $\Gamma^{(k)}$ for the rescaled configuration (7.2) is given as [52]

$$\lim_{N\to\infty} \frac{1}{N^2}\Gamma^{(k)} = \frac{1}{k^2}\left(\frac{1}{8}\,\tilde{\beta}^4 - \frac{1}{6}\,\tilde{\alpha}\,\tilde{\beta}^3\right) + \log\tilde{\beta} - \log k\,. \tag{7.3}$$

Similarly to the single fuzzy sphere case ($k = 1$), the effective action (7.3) has a local minimum at

$$\tilde{\beta} = \frac{1}{4}\,\tilde{\alpha}\left(1 + \sqrt{1+\delta^{(k)}} + \sqrt{2 - \delta^{(k)} + \frac{2}{\sqrt{1+\delta^{(k)}}}}\right), \tag{7.4}$$

where

$$\delta^{(k)} = 4\,\tilde{\alpha}^{-\frac{4}{3}}\left[\left(1+\sqrt{1-\frac{512\,k^2}{27\,\tilde{\alpha}^4}}\right)^{\frac{1}{3}} + \left(1-\sqrt{1-\frac{512\,k^2}{27\,\tilde{\alpha}^4}}\right)^{\frac{1}{3}}\right]. \tag{7.5}$$

The local minimum exists if and only if $\tilde{\alpha} > \tilde{\alpha}_{\rm cr}^{(k)} = \sqrt[4]{\frac{512\,k^2}{27}}$.

In figure 5 we plot the all order result for the free energy for $k = 1, 2, \ldots, 6$, which is obtained by evaluating the effective action (7.3) at its extremum (7.4). We find that the single fuzzy sphere ($k = 1$) always has lower free energy than the coinciding fuzzy spheres ($k \geq 2$). Thus we conclude that the dynamically generated gauge group is of rank one even if we take account of the higher order contributions. This conclusion is consistent with our observations in the Monte Carlo simulation [52].

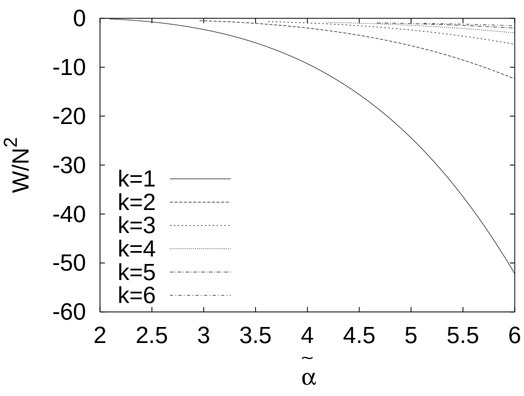

Figure 5: The all order result of the free energy for k-coincident fuzzy spheres ($k = 1, 2, \cdots, 6$) is plotted against $\tilde{\alpha}$.

8. Summary

In this paper we have clarified some peculiar aspects of the perturbative calculation around fuzzy sphere solutions in matrix models at large N. By direct two-loop calculation we have shown that the one-loop dominance, which holds for the effective action, does not hold for observables in general. However, we can obtain all order results for observables from one-loop calculation by shifting the center of expansion to the quantum solution, which extremizes the effective action. The validity of this method has been demonstrated by comparison with the direct two-loop calculation for the free energy and $\langle S \rangle$. We have also shown that the all order results for various observables reproduce nicely the behavior of our Monte Carlo data near the critical point.

From the all order results we have found that the perturbative expansion around the fuzzy sphere has a finite radius of convergence. We recall that similar phenomena occur in exactly solvable matrix models in the planar large-N limit due to the exponential — rather than factorial — growth of the number of planar diagrams. In the present case the convergence property can be attributed to the one-loop dominance of the effective action.

The conclusions listed above should also hold for other matrix models, which incorporate higher-dimensional fuzzy manifolds.

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A. Details of the two-loop calculation

In this section we present the details of the two-loop calculation. In particular we explain how eqs. (3.19) and (3.21) are obtained.

Let us note first that the kinetic term (3.6) can be written as

$$S_{\rm kin} = N \, {\rm tr} \left[\frac{1}{2} \, \tilde{A}_\mu (\mathcal{P}_\lambda)^2 \tilde{A}_\mu + \bar{c} \, (\mathcal{P}_\lambda)^2 c \right] , \tag{A.1}$$

where $\mathcal{P}_\mu$ is an operator acting on a $N \times N$ matrix M as

$$\mathcal{P}_\mu M = [X_\mu , M] \tag{A.2}$$

with X_μ being the fuzzy sphere solution (2.3). The operator $(\mathcal{P}_\lambda)^2$ can be diagonalized by the so-called "matrix spherical harmonics" Y_{lm} $(0 \le l \le N-1, \, -l \le m \le l)$, which form a complete basis in the space of $N \times N$ matrices satisfying

$$\frac{1}{N} \, {\rm tr} \left(Y_{lm}^\dagger Y_{l'm'} \right) = \delta_{ll'} \, \delta_{mm'} \, , \tag{A.3}$$

$$Y_{lm}^\dagger = (-1)^m \, Y_{l,-m} \, . \tag{A.4}$$

Similarly to the usual spherical harmonics, they also possess the properties such as

$$\begin{aligned}
[L_+, Y_{lm}] &= \sqrt{(l-m)(l+m+1)}\, Y_{l,m+1}\,, \\
[L_-, Y_{lm}] &= \sqrt{(l+m)(l-m+1)}\, Y_{l,m-1}\,, \\
[L_3, Y_{lm}] &= m\, Y_{lm}\,,
\end{aligned} \tag{A.5}$$

where $L_\pm = L_1 \pm iL_2$. From these properties we find that Y_{lm} is an eigenvector of the operator $(\mathcal{P}_\lambda)^2$, namely

$$(\mathcal{P}_\lambda)^2\, Y_{lm} = \alpha^2\, l\,(l+1)\, Y_{lm}\,. \tag{A.6}$$

By expanding the matrices $\tilde{A}_\mu$, c and $\bar{c}$ as

$$\tilde{A}_\mu = \sum_{l=1}^{N-1}\sum_{m=-l}^{l} \tilde{A}_{\mu lm} Y_{lm}\,, \qquad c = \sum_{l=1}^{N-1}\sum_{m=-l}^{l} c_{lm} Y_{lm}\,, \qquad \bar{c} = \sum_{l=1}^{N-1}\sum_{m=-l}^{l} \bar{c}_{lm} Y_{jm}\,, \tag{A.7}$$

the propagators can be brought into the diagonal form

$$\langle \tilde{A}_{\mu lm} \tilde{A}_{\nu l'm'} \rangle_0 = \frac{1}{N\alpha^2}\frac{(-1)^m}{l\,(l+1)}\delta_{\mu\nu}\delta_{l,l'}\delta_{m,-m'}\,, \tag{A.8}$$

$$\langle c_{lm} \bar{c}_{l'm'} \rangle_0 = \frac{1}{N\alpha^2}\frac{(-1)^m}{l\,(l+1)}\delta_{l,l'}\delta_{m,-m'}\,, \tag{A.9}$$

where the symbol $\langle\,\cdot\,\rangle_0$ represents a VEV using only the kinetic term S_{kin} in eq. (3.5).

In the two-loop calculation, we use the identity

$$\mathrm{tr}\Big(Y_{l_1m_1} Y_{l_2m_2} Y_{l_3m_3}\Big) = (-1)^{N-1}\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{Bmatrix}, \tag{A.10}$$

where L is defined by $N = 2L+1$, and Wigner's $(3j)$ and $\{6j\}$ symbols are given explicitly [57] by Racah's formula as

$$\begin{aligned}
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{l_1-l_2-m_3}\sqrt{\Delta(l_1l_2l_3)}\,\times \\
&\times\sqrt{(l_1-m_1)!\,(l_1+m_1)!\,(l_2-m_2)!\,(l_2+m_2)!\,(l_3-m_3)!\,(l_3+m_3)!}\,\times \\
&\times\sum_t (-1)^t \Big\{t!\,(t-l_2+l_3+m_1)!\,(t-l_1-m_2+l_3)!\,(l_1+l_2-l_3-t)!\,\times \\
&\qquad\qquad \times (l_1-m_1-t)!\,(l_2+m_2-t)!\Big\}^{-1}.
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\begin{Bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{Bmatrix} &= \sqrt{\Delta(l_1l_2l_3)\,\Delta(l_1m_2m_3)\,\Delta(m_1l_2m_3)\,\Delta(m_1m_2l_3)}\,\times \\
&\times\sum_t (-1)^t (t+1)!\Big\{(t-l_1-l_2-l_3)!\,(t-l_1-m_2-m_3)!\,\times \\
&\qquad\qquad \times (t-m_1-l_2-m_3)!(t-m_1-m_2-l_3)!\Big\}^{-1}\times \\
&\times\Big\{(l_1+l_2+m_1+m_2-t)!(l_2+l_3+m_2+m_3-t)!\,\times \\
&\quad \times (l_3+l_1+m_3+m_1-t)!\Big\}^{-1},
\end{aligned} \tag{A.12}$$

where

$$\Delta(abc) = \frac{(a+b-c)!\,(b+c-a)!\,(c+a-b)!}{(a+b+c+1)!}\,. \tag{A.13}$$

The sum of t is taken over all positive integers such that no factorial has a negative argument. If we have a negative argument in the factorial elsewhere, the $(3j)$ and $\{6j\}$ symbols are defined to be zero. From (A.10) we also obtain the formula

$$[Y_{l_1,m_1}, Y_{l_2,m_2}] = \sum_{l_3=1}^{N-1}\sum_{m_3=-l_3}^{l_3} f^{l_3,m_3}_{l_1,m_1,l_2,m_2} Y_{l_3,m_3}\,, \tag{A.14}$$

$$\begin{aligned} f^{l_3,m_3}_{l_1,m_1,l_2,m_2} &= (-1)^{m_3}(-1)^{N-1}\{1-(-1)^{l_1+l_2+l_3}\}\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}\,\times \\ &\quad\times \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{Bmatrix}. \end{aligned} \tag{A.15}$$

With the help of these formulae, the contribution of the 1PI two-loop diagrams (a)–(d) to the coefficient $w_2(N)$ can be calculated as [27]

$$w_2^{(\mathrm{a})}(N) = -\frac{3}{N}F_1(N)\,, \tag{A.16}$$

$$w_2^{(\mathrm{b})}(N) = \frac{2}{N}\Big\{F_1(N) - F_2(N)\Big\}\,, \tag{A.17}$$

$$w_2^{(\mathrm{c})}(N) = \frac{4}{N}F_3(N)\,, \tag{A.18}$$

$$w_2^{(\mathrm{d})}(N) = \frac{1}{N}F_1(N)\,, \tag{A.19}$$

where the functions $F_1(N)$, $F_2(N)$ and $F_3(N)$ are given explicitly as

$$\begin{aligned} F_1(N) &= \frac{1}{N}\sum_{l_1,l_2=1}^{N-1}\frac{(2l_1+1)(2l_2+1)}{l_1(l_1+1)l_2(l_2+1)} - \\ &\quad - \sum_{l_1,l_2=1}^{N-1}(-1)^{l_1+l_2}\frac{(2l_1+1)(2l_2+1)}{l_1(l_1+1)l_2(l_2+1)}\begin{Bmatrix} L & L & l_2 \\ L & L & l_1 \end{Bmatrix}, \end{aligned} \tag{A.20}$$

$$\begin{aligned} F_2(N) &= \sum_{l_1,l_2,l_3=1}^{N-1}\sum_{m_1,m_2,m_3}\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}\{1-(-1)^{l_1+l_2+l_3}\}\begin{Bmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{Bmatrix}^2 \times \\ &\quad\times\Bigg[\frac{1}{2}\sqrt{(l_1-m_1)(l_1+m_1+1)(l_2+m_2)(l_2-m_2+1)}\,\times \\ &\qquad\times\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1+1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2+1 & m_3 \end{pmatrix} + \\ &\qquad+\frac{1}{2}\sqrt{(l_1+m_1)(l_1-m_1+1)(l_2-m_2)(l_2+m_2+1)}\,\times \\ &\qquad\times\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1-1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2-1 & m_3 \end{pmatrix} + m_1 m_2\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2\Bigg], \end{aligned} \tag{A.21}$$

$$F_3(N) = \sum_{l_1,l_2,l_3=1}^{N-1} \left\{1-(-1)^{l_1+l_2+l_3}\right\} \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)} \begin{Bmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{Bmatrix}^2 . \tag{A.22}$$

Summing up the contributions (A.16)–(A.19) and using the identity $F_1(N) = -2F_2(N)$, we obtain eq. (3.19).

Now let us calculate the contributions from the 1PR diagrams (f)–(h). We first evaluate the tadpole obtained by contracting two $\tilde{A}$'s in the second term in (3.7). It is given as

$$\begin{aligned} T_A &= -N\alpha \left\{ \mathrm{tr}\left([\tilde{A}_\mu, \tilde{A}_\nu][L_\mu, \tilde{A}_\nu]\right) + tr\left([\tilde{A}_\mu, \tilde{A}_\nu][L_\mu, \tilde{A}_\nu]\right) + \mathrm{tr}\left([\tilde{A}_\mu, \tilde{A}_\nu][L_\mu, \tilde{A}_\nu]\right) \right\} \\ &= -N\alpha \sum_{l_1,l_2,l_3=1}^{N-1} \sum_{m_1,m_2,m_3} \mathrm{tr}([Y_{l_1m_1}, Y_{l_2m_2}][L_\mu, Y_{l_3m_3}]) \times \\ &\quad \times \left(\tilde{A}_{\mu l_1 m_1} \tilde{A}_{\nu l_2 m_2} \tilde{A}_{\nu l_3 m_3} - \tilde{A}_{\nu l_1 m_1} \tilde{A}_{\mu l_2 m_2} \tilde{A}_{\nu l_3 m_3} + \tilde{A}_{\mu l_1 m_1} \tilde{A}_{\nu l_2 m_2} \tilde{A}_{\nu l_3 m_3} \right) \\ &= -\frac{1}{\alpha} \sum_{l_1,l_2=1}^{N-1} \sum_{m_1,m_2} \tilde{A}_{\mu l_1 m_1} \frac{(-1)^{m_2}}{l_2(l_2+1)} \times \\ &\quad \times \{2\,\mathrm{tr}([Y_{l_1m_1}, Y_{l_2m_2}][L_\mu, Y_{l_2-m_2}]) + \mathrm{tr}([Y_{l_2m_2}, Y_{l_2-m_2}][L_\mu, Y_{l_1m_1}])\} . \end{aligned} \tag{A.23}$$

The second term of (A.23) vanishes since

$$\begin{aligned} 0 &= \mathrm{tr}([L_\mu, Y_{l_1m_1}[Y_{l_2m_2}, Y_{l_2-m_2}]]) \\ &= \mathrm{tr}([L_\mu, Y_{l_1m_1}][Y_{l_2m_2}, Y_{l_2-m_2}]) + \\ &\quad + \mathrm{tr}(Y_{l_1m_1}[[L_\mu, Y_{l_2m_2}], Y_{l_2-m_2}]) + \mathrm{tr}(Y_{l_1m_1}[Y_{l_2m_2}, [L_\mu, Y_{l_2-m_2}]]) \\ &= \mathrm{tr}([L_\mu, Y_{l_1m_1}][Y_{l_2m_2}, Y_{l_2-m_2}]) + \\ &\quad + \mathrm{tr}(Y_{l_1m_1}[[L_\mu, Y_{l_2m_2}], Y_{l_2-m_2}]) - \mathrm{tr}(Y_{l_1m_1}[[L_\mu, Y_{l_2-m_2}], Y_{l_2m_2}]) , \end{aligned} \tag{A.24}$$

where the second and third terms in the last line cancel each other. Similarly we evaluate the tadpole obtained by contracting c and $\bar{c}$ in the fourth term in (3.7). It is given as

$$\begin{aligned} T_{\mathrm{gh}} &= -\alpha N \sum_{l_1,l_2,l_3=1}^{N-1} \sum_{m_1,m_2,m_3} \bar{c}_{l_2m_2} \tilde{A}_{\mu l_1 m_1} c_{l_3m_3}\, \mathrm{tr}([L_\mu, Y_{l_2m_2}][Y_{l_1m_1}, Y_{l_3m_3}]) \\ &= \frac{1}{\alpha} \sum_{l_1,l_2=1}^{N-1} \sum_{m_1,m_2} \tilde{A}_{\mu l_1 m_1} \frac{(-1)^{m_2}}{l_2(l_2+1)} \mathrm{tr}([L_\mu, Y_{l_2-m_2}][Y_{l_1m_1}, Y_{l_2m_2}]) . \end{aligned} \tag{A.25}$$

Thus we find that

$$T_{\mathrm{gh}} = -\frac{1}{2} T_A . \tag{A.26}$$

By contracting two $\tilde{A}$'s in the product of two tadpoles, we obtain the contribution from the diagrams (f)–(h) to the coefficient $w_2(N)$, which we denote as $w_2^{(\mathrm{f})}(N)$, $w_2^{(\mathrm{g})}(N)$ and $w_2^{(\mathrm{h})}(N)$, respectively. We obtain, for instance,

$$w_2^{(\mathrm{f})}(N) = 2\alpha^2 \sum_{l_1,l_2,l_3,l_4=1}^{N-1} \sum_{m_1,m_2,m_3,m_4} \tilde{A}_{\mu l_1 m_1} \tilde{A}_{\nu l_4 m_4} \frac{(-1)^{m_2+m_3}}{l_2(l_2+1)l_3(l_3+1)} \times$$

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$$
\begin{aligned}
&\times \mathrm{tr}([Y_{l_1 m_1}, Y_{l_2 m_2}][L_\mu, Y_{l_2 -m_2}])\ \mathrm{tr}([Y_{l_4 m_4}, Y_{l_3 m_3}][L_\nu, Y_{l_3 -m_3}]) \\
= &\frac{2}{N} \sum_{l_1,l_2,l_3=1}^{N-1} \sum_{m_1,m_2,m_3} \frac{(-1)^{m_1+m_2+m_3}}{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)} \times \\
&\times \mathrm{tr}([Y_{l_1 m_1}, Y_{l_2 m_2}][L_\mu, Y_{l_2 -m_2}])\ \mathrm{tr}([Y_{l_1 -m_1}, Y_{l_3 m_3}][L_\mu, Y_{l_3 -m_3}]) \\
= &\frac{2}{N} F_4(N)\,,
\end{aligned}
\tag{A.27}
$$

where we have defined

$$
\begin{aligned}
F_4(N) = &\sum_{l_1,l_2,l_3=1}^{N-1} \sum_{m_1,m_2,m_3} \frac{(2l_1+1)(2l_2+1)(2l_3+1)(-1)^{m_1+m_2+m_3}}{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)} \times \\
&\times (2 - 2(-1)^{l_1}) \begin{Bmatrix} l_1 & l_2 & l_2 \\ L & L & L \end{Bmatrix} \begin{Bmatrix} l_1 & l_3 & l_3 \\ L & L & L \end{Bmatrix} \times \\
&\times \Bigg[m_2 m_3 \begin{pmatrix} l_1 & l_2 & l_2 \\ m_1 & m_2 & -m_2 \end{pmatrix} \begin{pmatrix} l_1 & l_3 & l_3 \\ -m_1 & m_3 & -m_3 \end{pmatrix} + \\
&\quad + \frac{1}{2} \sqrt{(l_2+m_2)(l_2-m_2+1)(l_3-m_3)(l_3+m_3+1)} \times \\
&\quad \times \begin{pmatrix} l_1 & l_2 & l_2 \\ m_1 & m_2 & -m_2+1 \end{pmatrix} \begin{pmatrix} l_1 & l_3 & l_3 \\ -m_1 & m_3 & -m_3-1 \end{pmatrix} + \\
&\quad + \frac{1}{2} \sqrt{(l_2-m_2)(l_2+m_2+1)(l_3+m_3)(l_3-m_3+1)} \times \\
&\quad \times \begin{pmatrix} l_1 & l_2 & l_2 \\ m_1 & m_2 & -m_2-1 \end{pmatrix} \begin{pmatrix} l_1 & l_3 & l_3 \\ -m_1 & m_3 & -m_3+1 \end{pmatrix} \Bigg] .
\end{aligned}
\tag{A.28}
$$

Due to the relation (A.26), the contributions from the other diagrams are readily obtained as

$$
w_2^{(\mathrm{g})}(N) = \frac{1}{2N} F_4(N)\,, \tag{A.29}
$$

$$
w_2^{(\mathrm{h})}(N) = -\frac{2}{N} F_4(N)\,. \tag{A.30}
$$

Summing up all the contributions, we obtain eq. (3.21).

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